The Multi-Industry Trade Model with IO Linkages

International Trade (PhD), Spring 2023

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Overview

- This lecture introduces inout-output (IO) linkages into the multi-industry trade model.
- For exposition, we abstract from scale economies (i.e., $\psi_k = 0$, $\forall k$)
- Main implications
  - IO linkages magnify the gains from trade
  - IO linkages amplify the cost of distortive wedges (*e.g.*, markups, tariffs)
- **References:**
  - *Costinot & Rodriguez-Clare (2014, Section 3.4)*
  - *Caliendo & Parro (2014)*: application to NAFTA

Environment

- $i,\ n = 1, ..., N$ countries supplying differentiated variaties
- $k = 1, ..., K$ industries
- Perfect competition $\longrightarrow$ no entry-driven scale economies
- Labor is the only primary factor of production.
- Country i is endowed with L_i (inelastically-supplied) units of labor.

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- Country i is endowed with L_i (inelastically-supplied) units of labor.
- **Every product variety can be used as either a final consumption good or an intermediate input good.**

On Overview of the Product Space

- Product variaties are differentiated by country of origin *à la* Armington.
- Good in, k (*origin* $i \times$ *destination* $n \times$ *industry* k) can be used as a
  1. final consumption good
  2. intermediate input for production in various industries

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  2. intermediate input for production in various industries
- **Example:** A good sold from Japan (i) to the US (n) in the auto-industry (k) can be used for private consumption or as an input in transportation services.

Armington Demand for Final Goods

The representative consumer in country i has a Cobb-Douglas–CES utility function over goods sourced from different origin countries:

$$U_i\left(\mathbf{C}_{1i}, \dots, \mathbf{C}_{Ni}\right) = \prod_{k=1}^{K}\left[\sum_{n=1}^{N} C_{ni,k}^{\frac{\sigma_k - 1}{\sigma_k}}\right]^{\frac{\sigma_k}{\sigma_k - 1}\beta_{i,k}}$$

- ni, k indexes *origin* n $\times$ *destination* i $\times$*industry* k
- $\sigma_k \geq 1$ is the inter-national elasticity of substitution.
- $\beta_{i,k}$ is country i's (constant) share of final consumption on industry k goods.

Demand for Final Goods

- The representative consumer maximizes utility given prices (P) and net income (Y):

$$\max_{\mathbf{C}_i} U_i(\mathbf{C}_{1i}, \ldots, \mathbf{C}_{Ni}) \qquad s.t. \qquad \sum_{k}^{K} \sum_{n=1}^{N} P_{ni,k} C_{ni,k} \leq Y_i \qquad (\mathbf{CP})$$

- The CES demand function implied by (CP):

$$\underbrace{\lambda^{\mathcal{C}}_{ni,k} \equiv \frac{P_{ni,k} C_{ni,k}}{\beta_{i,k} Y_i}}_{\text{expenditure share}} = \left(\frac{P_{ni,k}}{P^{\mathcal{C}}_{i,k}} \right)^{1-\sigma_k}, \qquad \text{where} \qquad \underbrace{P^{\mathcal{C}}_{i,k} = \left[\sum_{n=1}^{N} P^{1-\sigma_k}_{ni,k} \right]^{\frac{1}{1-\sigma_k}}}_{\text{CES price index}}$$

Supply: Production Function

Production combines labor (L), and intermediate inputs for various industries (I_g):

$$Q_{i,k} \sim \sum_n \tau_{in,k} Q_{in,k} = \varphi_{i,k} \left(\frac{L_{i,k}}{1-\alpha_{i,k}} \right)^{1-\alpha_{i,k}} \prod_{g=1}^{K} \left(\frac{I_{i,g}}{\alpha_{i,gk}} \right)^{\alpha_{i,gk}}$$

- $Q_{in,k} = C_{in,k} + I_{in,k}$ (total output = final goods + intermediate inputs)
- $I_{i,g}$ is a composite CES input consisting of industry g goods

$$I_{i,g} = \left[I_{1i,g}^{\frac{\tilde{\sigma}g-1}{\tilde{\sigma}g}} + ... + I_{Ni,g}^{\frac{\tilde{\sigma}g-1}{\tilde{\sigma}g}} \right]^{\frac{\tilde{\sigma}g}{\tilde{\sigma}g-1}}$$

- $\alpha_{i,kg}$ is the share of industry g inputs in production ($\alpha_{i,k} \equiv \sum_g \alpha_{i,gk}$)

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Key assumption: $\tilde{\sigma}_k = \sigma_k \longrightarrow P^{\mathcal{I}}_{i,k} = P^{\mathcal{C}}_{i,k}, \quad \lambda^{\mathcal{I}}_{in,k} = \lambda^{\mathcal{C}}_{in,k}$

Supply: Prices and Input Expenditure

- Perfect competition + cost minimization imply

$$P_{in,k} = \frac{\tau_{in,k}}{\varphi_{i,k}} w_i^{1-\alpha_{i,k}} \prod_{g=1}^{K} P_{i,g}^{\alpha_{i,gk}}$$

- Total expenditure on intermediate inputs from industry g

$$E_{i,g}^{\mathcal{I}} \equiv P_{i,g} I_{i,g} = \sum_{k=1}^{K} \alpha_{i,gk} R_{i,k}$$

where $R_{i,k}$ is *gross* revenue collected by origin i–industry k:

$$R_{i,k} = \sum_{n=1}^{N} P_{in,k} Q_{in,k} \sim P_{ii,k} Q_{i,k}$$

A Summary of Aggregate Demand and Supply

- Share of expenditure on variety *in*, *k* (*final* + *intermediate*)

$$\lambda_{in,k} = \frac{P_{in,k}^{1-\sigma_k}}{\sum_{j=1}^{N} P_{jn,k}^{1-\sigma_k}}$$

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- Country *i*'s gross revenue from industry *k* sales:

$$R_{i,k} = \sum_{n=1}^{N} \lambda_{in,k} E_{n,k}$$

- Country *n*'s goiss expenditure on industry *k* goods

$$E_{n,k} = \underbrace{\beta_{n,k} Y_n}_{\text{final goods}} + E_{n,k}^{\mathcal{I}}$$

A Summary of Aggregate Demand and Supply

- Share of expenditure on variety in, k (*final* + *intermediate*)

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$$E_{n,k} = \beta_{n,k} w_n L_n + \sum_{g=1}^{K} \left(\alpha_{n,kg} R_{n,g}\right)$$

General Equilibrium

For a given choice of parameters, equilibrium consists of price indexes, $\mathbf{P} \equiv \{P_{i,k}\}$, wage rates, $\mathbf{w} \equiv \{w_i\}$, and industry-level gross expenditure and sales, $\{E_{i,k}, R_{i,k}\}_{i,k}$, such that

$$\begin{cases} P_{i,k} = \sum_{n=1}^{N} \left[P_{ni,k}\left(w_n, \mathbf{P}_n\right)^{-\epsilon_k} \right]^{-\frac{1}{\epsilon_k}} & (\forall i, k) \\ R_{i,k} = \sum_{n=1}^{N} \lambda_{in,k}\left(\mathbf{w}, \mathbf{P}\right) E_{n,k} & (\forall i, k) \\ E_{i,k} = \beta_{i,k} w_i L_i + \sum_{g=1}^{K} \left(\alpha_{i,kg} R_{i,g}\right) & (\forall i, k) \\ w_i L_i = \sum_{k=1}^{K} (1 - \alpha_{i,k}) R_{i,k} & (\forall i) \end{cases}$$

where variety-specific prices and expenditure shares are

$$\begin{cases} P_{in,k}\left(w_i, \mathbf{P}_i\right) = \frac{\tau_{in,k}}{\varphi_{i,k}} w_i^{1-\alpha_{i,k}} \prod_{g=1}^{K} P_{i,g}^{\alpha_{i,gk}} & (\forall i, k) \\ \lambda_{in,k}\left(\mathbf{w}, \mathbf{P}\right) = \frac{P_{in,k}(w_i, \mathbf{P}_i)^{-\epsilon_k}}{\sum_{j=1}^{N} P_{jn,k}\left(w_j, \mathbf{P}_j\right)^{-\epsilon_k}} & (\forall i, j, k) \end{cases}$$

Growth Accounting: Open Economy with IO Linkages

- We want to characterize the welfare effects of a technical shock to aggregate productivity , $\{\mathrm{d}\ln\varphi_{i,k}\}_{i,k}$, and iceberg trade costs $\{\mathrm{d}\ln\tau_{in,k}\}_{i,n,k}$.
- For homothetic preferences (in general) the welfare effects can be specified as

$$\mathrm{d}\ln W_i = \mathrm{d}\ln Y_i - \sum_{k=1}^{K}\sum_{n=1}^{N}\lambda^{\mathcal{C}}_{ni,k}\beta_{i,k}\mathrm{d}\ln P_{ni,k}$$

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$$\mathrm{d}\ln W_i = \mathrm{d}\ln Y_i - \sum_{k=1}^{K}\sum_{n=1}^{N}\lambda_{ni,k}\beta_{i,k}\mathrm{d}\ln P_{ni,k}$$

- We can simplify the above expression by appealing to the CES demand structure:

$$\mathrm{d}\ln\lambda_{ni,k} - \mathrm{d}\ln\lambda_{ii,k} = -\epsilon_k\left(\mathrm{d}\ln P_{ni,k} - \mathrm{d}\ln P_{ii,k}\right)$$

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$$\mathrm{d}\ln P_{ni,k} = \mathrm{d}\ln P_{ii,k} - \frac{1}{\epsilon_k}(\mathrm{d}\ln \lambda_{ni,k} - \mathrm{dln}\lambda_{ii,k})$$

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- We want to characterize the welfare effects of a technical shock to aggregate productivity , $\{d \ln \varphi_{i,k}\}_{i,k}$, and iceberg trade costs $\{d \ln \tau_{in,k}\}_{i,n,k}$.
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$$d\ln W_i = d \ln Y_i - \sum_{k=1}^{K} \sum_{n=1}^{N} \lambda_{ni,k} \beta_{i,k} d \ln P_{ni,k}$$

- We can simplify the above expression by appealing to the CES demand structure:[1]

$$d \ln P_{ni,k} = d \ln P_{ii,k} - \frac{1}{\epsilon_k} (d \ln \lambda_{ni,k} - d\ln \lambda_{ii,k})$$

[1] CES preferences ensure that $\epsilon_k \equiv \frac{\partial \ln(\lambda_{ni,k}/\lambda_{ii,k})}{\partial \ln(P_{ni,k}/P_{ii,k})}$ is a constant parameter. The above equation, however, holds *non-parametrically* if we treat ϵ_k as a local (and possibly variable) elasticity.

Growth Accounting: Open Economy with IO Linkages

- Plugging our earlier expression for $d\ln P_{ni,k}$ into the welfare equation yields

$$
\begin{aligned}
d\ln W_i &= d\ln Y_i - \sum_{k=1}^{K}\sum_{n=1}^{N} \beta_{i,k}\lambda_{ni,k} d\ln P_{ni,k} \\
&= d\ln Y_i - \sum_{k} \beta_{i,k} d\ln P_{ii,k} \; + \; \sum_{k}\sum_{n} \left[\frac{1}{\epsilon_k} \beta_{i,k}\lambda_{ni,k} \left(d\ln \lambda_{ni,k} - d\ln \lambda_{ii,k} \right) \right]
\end{aligned}
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- Appealing to adding up constraints, $\begin{cases}\sum_n \lambda_{ni,k} d\ln\lambda_{ni,k} = 0 \\ \sum_n \lambda_{ni,k} = 1\end{cases}$, the last line yields

$$d\ln W_i = d\ln Y_i - \sum_k\left[\beta_{i,k}\left(d\ln P_{ii,k} - \frac{1}{\epsilon_k} d\ln\lambda_{ii,k}\right)\right]$$

Growth Accounting: Open Economy with IO Linkages

- Since $P_{ii,k} = \varphi_{i,k}^{-1} w_i^{1-\alpha_{i,k}} \prod_{g=1}^{K} P_{i,g}^{\alpha_{i,gk}}$, we can specify the change in domestic prices as

$$\begin{aligned} \mathrm{d}\ln P_{ii,k} &= -\mathrm{d}\ln \varphi_{i,k} + (1-\alpha_{i,k})\,\mathrm{d}\ln w_i + \sum_g \alpha_{i,gk}\mathrm{d}\ln P_{i,g} \\ &= -\mathrm{d}\ln \varphi_{i,k} + (1-\alpha_{i,k})\,\mathrm{d}\ln w_i + \sum_g \sum_n \alpha_{i,gk}\lambda_{ni,g}\mathrm{d}\ln P_{ni,g} \end{aligned}$$

[2]The expression for $\mathrm{d}\ln P_{ii,k}$ holds also non-parametrically following Shephard's lemma.

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- CES demand implies $\mathrm{d}\ln P_{ni,k} = \mathrm{d}\ln P_{ii,k} - \frac{1}{\epsilon_k}\left(\mathrm{d}\ln \lambda_{in,k} - \mathrm{d}\ln \lambda_{ii,k}\right)$, which when plugged into the above equation delivers (similar to the previous slide)

$$\mathrm{d}\ln P_{ii,k} = -\mathrm{d}\ln \varphi_{i,k} + (1-\alpha_{i,k})\,\mathrm{d}\ln w_i + \sum_g \alpha_{i,gk} \left(\mathrm{d}\ln P_{ii,g} + \frac{1}{\epsilon_g} \mathrm{d}\ln \lambda_{ii,g} \right)$$

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Growth Accounting: Open Economy with IO Linkages

- Since $P_{ii,k} = \varphi_{i,k}^{-1} w_i^{1-\alpha_{i,k}} \prod_{g=1}^{K} P_{i,g}^{\alpha_{i,gk}}$, we can specify the change in domestic prices as

$$\begin{aligned} d \ln P_{ii,k} &= -d \ln \varphi_{i,k} + (1 - \alpha_{i,k})\, d \ln w_i + \sum_g \alpha_{i,gk} d \ln P_{i,g} \\ &= -d \ln \varphi_{i,k} + (1 - \alpha_{i,k})\, d \ln w_i + \sum_g \sum_n \alpha_{i,gk} \lambda_{ni,g} d \ln P_{ni,g} \end{aligned}$$

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$$d \ln P_{ii,k} = \underbrace{-d \ln \varphi_{i,k} + (1 - \alpha_{i,k})\, d \ln w_i}_{B_{i,k}} + \sum_g \alpha_{i,gk} \left(d \ln P_{ii,g} + \frac{1}{\epsilon_g} d \ln \lambda_{ii,g} \right)$$

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Growth Accounting: Open Economy with IO Linkages

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- The above equation can be represented in matrix form as ($\Lambda_{ii,g} \sim \frac{1}{\epsilon_g}\mathrm{d}\ln \lambda_{ii,g}$)

$$\mathrm{d}\ln \mathbf{P}_{ii} = \mathbf{B}_i + \mathbf{A}_i^{\mathsf{T}}\left(\mathrm{d}\ln \mathbf{P}_{ii} + \mathbf{\Lambda}_{ii}\right)$$

[2] The expression for $\mathrm{d}\ln P_{ii,k}$ holds also non-parametrically following Shephard's lemma.

Growth Accounting: Open Economy with IO Linkages

- Since $P_{ii,k} = \varphi_{i,k}^{-1} w_i^{1-\alpha_{i,k}} \prod_{g=1}^{K} P_{i,g}^{\alpha_{i,gk}}$, we can specify the change in domestic prices as[2]

$$\begin{aligned} \mathrm{d}\ln P_{ii,k} &= -\mathrm{d}\ln \varphi_{i,k} + (1-\alpha_{i,k})\,\mathrm{d}\ln w_i + \sum_g \alpha_{i,gk}\mathrm{d}\ln P_{i,g} \\ &= -\mathrm{d}\ln \varphi_{i,k} + (1-\alpha_{i,k})\,\mathrm{d}\ln w_i + \sum_g \sum_n \alpha_{i,gk}\lambda_{ni,g}\mathrm{d}\ln P_{ni,g} \end{aligned}$$

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$$\mathrm{d}\ln \mathbf{P}_{ii} = \left(\mathbf{I} - \mathbf{A}_i^{T}\right)^{-1}\mathbf{B}_i - \left(\mathbf{I} - \mathbf{A}_i^{T}\right)^{-1}\mathbf{A}_i^{T}\boldsymbol{\Lambda}_{ii}$$

[2] The expression for $\mathrm{d}\ln P_{ii,k}$ holds also non-parametrically following Shephard's lemma.

Growth Accounting: Open Economy with IO Linkages

- Denote by $\tilde{\mathbf{A}}_i = (\mathbf{I} - \mathbf{A}_i)^{-1}$ the Leontief inverse, with $\tilde{\alpha}_{i,kg}$ denoting entry (k, g) of $\tilde{\mathbf{A}}_i$.
- We use two properties of the Leontief inverse:

$$\tilde{\mathbf{A}}_i^{\mathrm{T}} = \left(\mathbf{I} - \mathbf{A}_i^{\mathrm{T}}\right)^{-1} \qquad \left(\mathbf{I} - \mathbf{A}_i^{\mathrm{T}}\right)^{-1} \mathbf{A}_i^{\mathrm{T}} = \tilde{\mathbf{A}}^{\mathrm{T}} - \mathbf{I}$$

- Appealing to these properties, our previously-derived expression for $\mathrm{d} \ln \mathbf{P}_{ii}$ implies

$$\mathrm{d} \ln P_{ii,k} = \sum_g \left[\tilde{\alpha}_{i,gk} \left(-\mathrm{d} \ln \varphi_{i,g} + (1 - \alpha_{i,g})\, \mathrm{d} \ln w_i + \frac{1}{\epsilon_g} \mathrm{d} \ln \lambda_{ii,g}\right)\right] - \frac{1}{\epsilon_k} \mathrm{d} \ln \lambda_{ii,k} \qquad (*)$$

Growth Accounting: Open Economy with IO Linkages

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$$\mathrm{d}\ln P_{ii,k} = \sum_g \left[\tilde{\alpha}_{i,gk}\left(-\mathrm{d}\ln \varphi_{i,g} + (1-\alpha_{i,g})\,\mathrm{d}\ln w_i + \frac{1}{\epsilon_g}\mathrm{d}\ln \lambda_{ii,g}\right)\right] - \frac{1}{\epsilon_k}\mathrm{d}\ln \lambda_{ii,k} \qquad (*)$$

- *Note*: absent IO linkages $\longrightarrow$ $\mathrm{d}\ln P_{ii,k} = -\mathrm{d}\ln \varphi_{i,k} + \mathrm{d}\ln w_i$

Growth Accounting: Open Economy with IO Linkages

- $Y_i = w_i L_i \xrightarrow{d\ln L_i = 0} d\ln Y_i = d\ln w_i \qquad (**)$
- Plugging Equations $(*)$ & $(**)$ into our earlier expression for $d\ln W_i$, yields

$$d\ln W_i = d\ln Y_i - \sum_k \left[\beta_{i,k}\left(d\ln P_{ii,k} + \frac{1}{\epsilon_k} d\ln \lambda_{ii,k}\right)\right]$$

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$$\mathrm{d}\ln W_i = \left(1 - \sum_{g,k}\left(1-\alpha_{i,g}\right)\tilde{\alpha}_{i,gk}\beta_{i,k}\right)\mathrm{d}\ln w_i - \sum_k \left[\beta_{i,k}\sum_g \tilde{\alpha}_{i,gk}\left(-\mathrm{d}\ln\varphi_{i,g} + \frac{1}{\epsilon_g}\mathrm{d}\ln\lambda_{ii,g}\right)\right]$$

Growth Accounting: Open Economy with IO Linkages

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Growth Accounting: Open Economy with IO Linkages

- $Y_i = w_i L_i \xrightarrow{d \ln L_i = 0} d \ln Y_i = d \ln w_i \qquad (**)$
- Plugging Equations $(*)$ & $(**)$ into our earlier expression for $\mathrm{d}\ln W_i$, yields

$$\mathrm{d}\ln W_i = \underbrace{\left(1 - \sum_{g,k} (1 - \alpha_{i,g})\, \tilde{\alpha}_{i,gk} \beta_{i,k}\right)}_{=0} \mathrm{d}\ln w_i - \sum_k \left[\beta_{i,k} \sum_g \tilde{\alpha}_{i,gk} \left(-\mathrm{d}\ln \varphi_{i,g} + \frac{1}{\epsilon_g} \mathrm{d}\ln \lambda_{ii,g}\right)\right]$$

Proposition 1: Consider a small shock to productivity, $\mathrm{d}\ln \varphi$, and trade costs, $\mathrm{d}\ln \tau$. The resulting welfare impact is

$$\mathrm{d}\ln W_i = \sum_g \sum_k \left[\beta_{i,k} \tilde{\alpha}_{i,gk} \mathrm{d}\ln \varphi_{i,g}\right] - \sum_g \sum_k \left[\beta_{i,k} \tilde{\alpha}_{i,kg} \frac{1}{\epsilon_g} \mathrm{d}\ln \lambda_{ii,g}\right]$$

where $\tilde{\alpha}_{i,gk}$ is entry (k, g) of the Leontief inverse and $\beta_{i,k}$ is the share of *consumption* expenditure on industry k goods.

Taking Stock

- The formulas derived for $d \ln W_i$ hold non-parametrically as long as preferences are homothetic and stable and production is constant-returns to scale.
- The CES and Cobb-Douglas parameterization allows us to use these local formulas to calculate the impacts of a *large* change in trade costs.

Taking Stock

- The formulas derived for $\mathrm{d}\ln W_i$ hold non-parametrically as long as preferences are homothetic and stable and production is constant-returns to scale.
- The CES and Cobb-Douglas parameterization allows us to use these local formulas to calculate the impacts of a *large* change in trade costs.
- For a closed economy the formula we derived reduces to Hulten (1978). In particular, setting $\mathrm{d}\ln\lambda_{ii,k}=0$, yields $\mathrm{d}\ln W_i=\sum_g\sum_k\beta_{i,k}\tilde{\alpha}_{i,gk}\mathrm{d}\ln\varphi_{i,g}$, which considering that $\sum_k\beta_{i,k}\tilde{\alpha}_{i,gk}=\frac{P_{i,g}Q_{i,g}}{Y_i}$, deliver Hulten's formula:

$$\mathrm{d}\ln W_i=\sum_g \underbrace{\frac{P_{ii,g}Q_{ii,g}}{Y_i}}_{\text{Domar weight}}\mathrm{d}\ln\varphi_{i,g}$$

The Gains From Trade under IO Linkages

- Define the gains from trade as the ex-post gains from trade openness relative to autarky ($\tau = \infty$)

$$\text{GT}_i \equiv \frac{W_i - W_i^A}{W_i} = 1 - \exp\left(- \int_{\tau}^{\infty} \mathrm{d} \ln W_i \right)$$

- Per Proposition 1, we can specify $\mathrm{d} \ln W_i$ in response to $\mathrm{d} \ln \tau$ (setting $\mathrm{d} \ln \varphi = 0$) as

$$\mathrm{d} \ln W_i = \sum_g \sum_k \left[\beta_{i,k} \tilde{\alpha}_{i,kg} \frac{1}{\epsilon_g} \mathrm{d} \ln \lambda_{ii,g} \right]$$

where $\tilde{\alpha}_{i,gk}$ are entries of the Leontief inverse and $\beta_{i,k}$ are *consumption* shares.

The Gains From Trade under IO Linkages

- Plugging $\mathrm{d}\ln W_i$ into the expression for GT_i and noting that transitioning to autarky amounts to raising $\lambda_{ii,k}$ from its factual level to $\lambda^A_{ii.k} = 1$, delivers

$$\mathrm{GT}_i = 1 - \exp\left(-\int_{\lambda_{ii,g}}^{1} \sum_{k,g} \beta_{i,k}\tilde{\alpha}_{i,kg}\frac{1}{\epsilon_g}\mathrm{d}\ln\lambda_{ii,g}\right)$$

$$1 - \exp\left(-\sum_{k,g}\left[\beta_{i,k}\tilde{\alpha}_{i,kg}\frac{1}{\epsilon_g}\int_{\lambda_{ii,g}}^{1}\mathrm{d}\ln\lambda_{ii,g}\right]\right)$$

$$= 1 - \exp\left(\sum_{k,g}\left[\beta_{i,k}\tilde{\alpha}_{i,kg}\frac{1}{\epsilon_g}\right]\ln\lambda_{ii,g}\right) = 1 - \prod_k\prod_g \lambda_{ii,g}^{\frac{\tilde{\alpha}_{i,kg}}{\epsilon_g}\beta_{i,k}}$$

Directions for Computing the Gains from Trade under IO Linkages

- **Step 1:** compile industry-level data for domestic expenditure shares, $\{\lambda_{ii,k}\}_k$, consumption shares, $\{\beta_{i,k}\}_k$, and trade elasticities, $\{\epsilon_g\}_g$.[3]

- **Step 2:** use the national-level I-O matrix, $\boldsymbol{A}_i \equiv \left[\alpha_{i,gk}\right]_{k,g}$, to compute the element of the *Leontief inverse:*

$$\left[\tilde{\alpha}_{i,gk}\right]_{k,g} = \left(\boldsymbol{I} - \boldsymbol{A}_i\right)^{-1}$$

- **Step 3:** plug data points obtained in Steps 1 and 2 into the gains from trade formula:

$$GT_i = 1 - \prod_{k=1}^{K} \prod_{g=1}^{K} \lambda_{ii,g}^{\frac{\tilde{\alpha}_{i,kg}}{\epsilon_g} \beta_{i,k}}$$

[3]The WIOD is the standard source for this type of data.

The Gains from Trade are Amplified by IO Linkages

	% GT	
	w/o IO Linakges	w/ IO Linakges
Ireland	8%	37.1%
Belgium	7.8%	54.6%
Germany	4.5%	21.6%
China	2.6%	11.5%
U.S.	1.8%	8.3%

Source: *Costinot & Rodriguez-Clare (2014)* based on data from the 2008 WIOD, which cover 16 industries.

Performing Counterfactuals using Exact Hat-Algebra

- Consider a possibly large shock to trade costs: $\left\{\hat{\tau}_{in,k}\right\}_{i,n}$
- The equilibrium responses, $\left\{\hat{Y}_i, \hat{P}_{i,k}, \hat{R}_{i,k}, \hat{E}_{i,k}\right\}$ can be obtained by solving the following system:

$$\begin{cases} \hat{P}_{i,k} = \left[\sum_{n=1}^{N} \lambda_{ni,k} \left(\hat{P}_{ni,k}\right)^{-\epsilon_k}\right]^{-\frac{1}{\epsilon_k}} & \forall\,(i,k) \\ \hat{R}_{i,k} R_{i,k} = \sum_{n=1}^{N} \hat{\lambda}_{in,k} \lambda_{in,k} \hat{E}_{n,k} E_{n,k} & \forall\,(i,k) \\ \hat{E}_{i,k} E_{i,k} = \beta_{i,k} \hat{Y}_i Y_i + \sum_{g=1}^{K} \left(\alpha_{i,kg} \hat{R}_{i,g} R_{i,g}\right) & \forall\,(i,k) \\ \hat{Y}_i Y_i = \sum_{k=1}^{K} (1-\alpha_{i,k}) \hat{R}_{i,k} R_{i,k} & \forall i \end{cases}$$

where the non-highlighted variables are data and $\hat{P}_{ni,k}$ and $\hat{\lambda}_{ni,k}$ are given by

$$\hat{P}_{ni,k} = \hat{\tau}_{ni,k} \left(\hat{Y}_n\right)^{1-\alpha_{i,k}} \prod_{g=1}^{K} \left(\hat{P}_{n,g}\right)^{\alpha_{i,gk}} \qquad \hat{\lambda}_{ni,k} = \left(\hat{P}_{ni,k} / \hat{P}_{i,k}\right)^{-\epsilon_k}$$

Measuring Welfare Effects

- Given the obtained solution $\left\{ \hat{Y}_i, \hat{P}_{i,k} \right\}_i$, we can calculate the change in welfare as

$$\%\Delta W_i = 100 \times \left(\frac{\hat{Y}_i}{\hat{P}_i} - 1 \right) \qquad \hat{P}_i = \prod_{n=1}^{N} \left(\hat{P}_{i,k} \right)^{\beta_{i,k}}$$

Measuring Welfare Effects

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- A similar approach can be applied to compute the impact of tariff reduction (albeit one must update the previous system to accommodate tariff revenues)

- **Notable Application:** Calinedo & Parro (2015) perform the note analysis to compute the welfare impacts of NAFTA-related tariff cuts:

$$\Delta W_{\text{MEX}} = 1.31\% \qquad \Delta W_{\text{CAN}} = -0.06\% \qquad \Delta W_{\text{USA}} = 0.08\%$$